Richardson's Marvellous Forecast

by

Peter Lynch

Irish Meteorological Service, Dublin.

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1. INTRODUCTION

While Vilhelm Bjerknes and his team were developing their synoptic models in Bergen, a radically different approach to forecasting was being pursued by Lewis Fry Richardson. Richardson's starting point was the system of fundamental physical principles governing atmospheric motion. He assembled the set of mathematical equations which represent these principles and formulated an approximate algebraic method of calculating their solution. Starting from the state of the atmosphere at a given time — the initial conditions — the method could be used to work out its future evolution.

Using the most complete set of observations available to him, Richardson applied his numerical method and calculated the changes in the pressure and winds at two points in central Europe. The results were something of a calamity: Richardson calculated a change in surface pressure over a six-hour period of 145 hPa, a totally unrealistic value. As Sir Napier Shaw remarked, the wildest guess would not have been wider of the mark! Despite the "glaring errors" in his forecast, Richardson was bold enough to publish his method and results in his remarkable *Weather Prediction by Numerical Process* (LFR; Richardson, 1922). This profound, and occasionally whimsical, book is a treasure-store of original and thought-provoking ideas and amply repays the effort required to read it.

The application of Richardson's forecasting method involved an enormous amount of numerical computation. Even the limited results he obtained cost him some two years of arduous calculation (Lynch, 1993). This work was carried out in the Champagne district of France where Richardson served as an ambulance driver during the Great War (Ashford, 1985). His dedication and tenacity in the dreadful conditions of the war are an inspiration to those of us who work in more genial conditions.

In this chapter the results obtained by Richardson will be examined and the causes of the errors in his forecast will be explained. It will be shown how a realistic forecast can be obtained by modifying the initial data. The study is based on the original observations for 20 May, 1910, originally compiled by Hugo Hergessel and analysed by Vilhelm Bjerknes. These are used to extend the table of values published by Richardson, to cover most of Europe. A numerical model is then constructed, keeping as close as possible to the method of Richardson, except for omission of minor physical processes. When the model is run with the extended data, the results are virtually identical to those of Richardson. In particular, an initial pressure tendency of 145 hPa in 6 hours is obtained at the central point, in agreement with Richardson. The tendency values are unrealistic, being generally about two orders of magnitude too large.

The reasons for the spurious tendencies will be discussed. They are essentially due

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to an imbalance between the pressure and wind fields resulting in large amplitude high frequency gravity wave oscillations. The 'cure' is to modify the analysis so as to restore balance; this process is called *initialization.* An initialization method based on a digital filter will be outlined, and its application to Richardson's problem described. The forecast tendency from the modified data yields reasonable results. In particular, the tendency at the central point is reduced to 3 hPa per 6 hours — a realistic value! The chapter will conclude with some speculations about *what-might-have-been* had Richardson been able to initialize his data.

2. OBSERVATIONS AND INITIAL FIELDS

2.1 Observational Data

The forecast made by Richardson was based on "one of the most complete sets of observations on record" (LFR, p. 181). During the first decades of the century observations of conditions at the earth's surface were made on a regular basis, and daily surface weather maps were issued by several centres. Upper air observations were made only intermittently, typically for one or a few days each month, as agreed by the countries participating in the work of the International Commission for Scientific Aeronautics (ICSA). The data were compiled and published by the Meteorological Institute of Strasbourg, under the editorship of Hugo Hergessel, Director of the Institute and President of ICSA.

A detailed analysis of the aerological observations was undertaken by Vilhelm Bjerknes at the Geophysical Institute in Leipzig. He produced a publication series consisting of sets of charts of atmospheric conditions at ten standard pressure levels from 100 hPa to 1000 hPa. These charts provided Richardson with the data required for his arithmetical forecasting procedure. The initial date and time chosen by Richardson for his forecast was 20 May, 1910, 0700 UTC. For this time there were 12 soundings and 18 reports of upper level winds over western Europe.[1] The observations are tabulated in a synopsized form in Bjerknes (1914).

2.2 Preparation of the Initial Fields

Richardson chose to divide the atmosphere into five layers, centered approximately at pressures 900, 700, 500, 300 and 100 hPa. He divided each layer into boxes and assumed that the value of a variable in each box could be represented by its value at the central point; we refer to such points as grid-points. They were separated by $\Delta\lambda = 3.0^\circ$ in longitude and $\Delta\phi = 1.8^\circ$ in latitude. Richardson tabulated his initial values for a selection of points over central Europe. The area is shown on a map on p184 of LFR, and the values are given in his "Table of Initial Distribution" on p185.

In §9/1 of LFR, Richardson describes the various steps he took in preparing his initial data. He prepared the mass and wind analyses independently (univariate analysis). The initial fields used in the present study were obtained from the same source, but we did not follow precisely the method of Richardson; the procedure adopted is outlined below. In order that the geostrophic relationship should not be allowed to dominate the choice of values, the pressure and velocity analyses were performed separately (and by

[1] On 20 May 1910 the observation for Vienna comprised only winds; no pressures or temperatures were available, the registering balloon being recorded as *bis heute noch nicht gefunden.* It is unlikely to be found now.

two different people; see Acknowledgement).

The initial pressure fields were derived from Bjerknes' charts of geopotential height at 200, 400 600 and 800 hPa (his charts 6, 8, 10 and 12). A transparent sheet marked with the grid-points was super-imposed on each chart and the height at each point read off. Each level p_k corresponds to a standard height z_k with temperature T_k. Conversion from height z to pressure p was made using the simple formula

$$p = p_k \left(1 + \frac{z - z_k}{H_k}\right)$$

where $H_k = \mathcal{R}T_k/g$. Sea-level pressure values were extracted in the same way as heights, from Bjerknes' chart number 1. His values in mm Hg were converted to hectopascals by multiplication by 4/3. Then the surface pressure p_S was calculated from

$$p_S = p_{SEA} \left(1 - \frac{\gamma h}{T_0}\right)^{g/\gamma\mathcal{R}}$$

where h is orographic height at the point in question, and standard values $T_0 = 288\,\mathrm{K}$ and $\gamma = 0.0065\,\mathrm{K\,m^{-1}}$ were used for the surface temperature and vertical lapse-rate.

The initial values of momenta for each of the five layers are required. These were derived from the wind velocities at the intermediate levels 100, 300, 500, 700 and 900 hPa. The observed wind speeds and directions for each level were plotted on charts upon which isotachs and isogons were then drawn by hand. The grid-point values of speed and direction could now be read off. It was necessary to exercise a degree of imagination as the observational coverage was so limited. The wind values were converted to components u and v and the layer momenta U and V were defined by

$$U = \frac{\Delta p}{g} v \qquad V = \frac{\Delta p}{g} v$$

where Δp is the pressure across the layer (obtained in the pressure analysis).

The pressure, temperature and momentum values, at a selection of points in the centre of the domain, resulting from the re-analysis, are given in Table 1. The corresponding values obtained and used by Richardson, extracted from his "Table of Initial Distribution" (LFR, p185), are reproduced in Table 2. The stratospheric temperatures and orographic heights are also indicated (top and bottom numbers in each white block). To facilitate comparison, the orography values used by Richardson were used (where available) in the re-analysis.

There is reasonable agreement between the pressure and stratospheric temperature values in the two tables. In general, pressure differences are within one or two hectopascals. There is a notable exception at the point $(48.6^\circ\,\mathrm{N}, 5.0^\circ\,\mathrm{E})$, where the old and new values differ by 10 hPa. We will see below that Richardson's value at this point is suspect.

Table I: Initial Distribution, Re-analysed Values

	$5°E$	$8°E$	$11°E$	$14°E$	$17°E$
$54.0°N$			**106** **-228** **120** **-144** **0** **-81** **-97** **0** **-221** **81** 0		
$52.2°N$		**-62** **-138** **-133** **-135** **-155** 150	212 206 410 609 799 987 200	**-25** **-79** **-107** **-156** **-181** 100	
$50.4°N$	**-175** **208** **-292** **263** **-249** **174** **-118** **99** **-88** **51** 200	216 205 409 607 796 983 200	**-105** **-182** **-268** **-38** **-201** **-18** **-199** **73** **-127** **73** 400	212 206 410 608 798 976 300	**-126** **-218** **-167** **-213** **-155** **-130** **-214** **0** **-175** **82** 300
$48.6°N$	221 204 406 605 793 984 200	**-159** **-275** **-216** **-131** **-60** 400	214 206 410 608 796 961 400	**-131** **-205** **-147** **-129** **-81** 400	213 206 410 608 798 989 200
$46.8°N$		217 204 405 604 794 872 1200	**-208** **18** **-289** **0** **-172** **172** **-45** **64** **-32** **38** 1800	213 205 407 606 796 842 1500	
$45.0°N$			210 203 404 603 795 995 100		

Comparing the momenta in Tables 1 and 2, we see more significant discrepancies. Although the overall flow suggested by the momenta is similar in each case, point values are radically different from each other, with variations as large as the values themselves and occasional differences of sign. These dissimilarities arise partly from the different analysis procedures used, but mainly from the large margin of error involved in the interpolation from the very few observations to the grid-points. In repeating the forecast, we have simply replaced the re-analysed values of all fields by Richardson's original values at the few gridpoints where the latter are available. The values in Table 2 are thus the initial values for both Richardson's forecast and the forecasts described below.

3. THE FUNDAMENTAL EQUATIONS

The behaviour of the atmosphere is governed by the fundamental principles of conservation of mass, energy and momentum. These principles may be expressed in terms of differential equations. The idea of solving the equations to calculate future weather was propounded in a famous address by Bjerknes (1904). The first attempt to put this idea into practice was that of Richardson.

Richardson was careful not to make any unnecessary approximations, and he took account of several physical processes which had the most marginal effect on his forecast. He included in his equations many terms which are negligible; with the benefit of hindsight, we can omit most of these. We shall ignore all the effects of moisture and thermal forcing, and consider the adiabatic evolution of a dry atmosphere. One fundamental approximation was made by Richardson: the atmosphere is in a state of hydrostatic balance. This was an essential step, necessitated by the lack of observations of vertical velocity, and it enabled Richardson to derive his elegant diagnostic equation for this quantity.

We shall set out the basic equations as commonly used today, and then convert them to the form used by Richardson. Some of Richardson's notation is archaic and the modern equivalents will be used (a full Table of his notation is found in Ch. XII of LFR). The independent variables are latitude ϕ, longitude λ, height z and time t. Distances eastward and northward are denoted x and y. The dependent variables are the eastward, northward and upward components of velocity (u, v, w), pressure p, temperature T and density ρ.

3.1 The Primitive Equations.

The primitive equations may be found in standard texts on dynamic meteorology. The equations of motion are:

$$\frac{du}{dt} - \left(f + \frac{u\tan\phi}{a}\right)v + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

$$\frac{dv}{dt} + \left(f + \frac{u\tan\phi}{a}\right)u + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0$$

$$\frac{\partial p}{\partial z} + g\rho = 0$$

The Earth's radius is a, its angular velocity is Ω and $f = 2\Omega\sin\phi$ is the Coriolis

Table II: Initial Distribution, Richardson's Values

	$5°E$	$8°E$	$11°E$	$14°E$	$17°E$
$54.0°N$			**-65** **8** **127** **-104** **81** **-25** **-81** **0** **-198** **84** 0		
$52.2°N$		**-70** **-62** **-114** **-91** **-160** 150	214 205 409 609 798 988 200	**-160** **40** **-60** **-60** **-219** 100	
$50.4°N$	**-30** **-110** **-245** **300** **-223** **158** **-91** **87** **-18** **15** 200	212 205 408 607 795 983 200	**-56** **-18** **-146** **-62** **-95** **29** **-52** **58** **-110** **55** 400	214 205 409 609 798 976 300	**-100** **-32** **0** **-260** **-55** **-135** **-25** **48** **-190** **160** 300
$48.6°N$	214 203 405 604 793 974 200	**27** **-328** **-136** **-33** **48** 400	212 205 409 608 796 963 400	**0** **-166** **-95** **-19** **-65** 400	214 204 408 607 798 988 200
$46.8°N$		214 204 406 605 795 875 1200	**-50** **80** **-280** **41** **-175** **150** **-105** **80** **-155** **40** 1800	214 204 408 607 797 846 1500	
$45.0°N$			213 203 403 603 796 997 100		

parameter.

The continuity equation, expressing conservation of mass, is

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} - \frac{\rho v \tan\phi}{a} + \frac{\partial \rho w}{\partial z} = 0$$

(a small term $2\rho w/a$ has been dropped). In combination with this equation, the horizontal equations of motion may be written in flux form:

$$\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2}{\partial x} + \frac{\partial \rho u v}{\partial y} + \frac{\partial \rho u w}{\partial z} - \left(f + \frac{2u\tan\phi}{a}\right)\rho v + \frac{\partial p}{\partial x} = 0$$

$$\frac{\partial \rho v}{\partial t} + \frac{\partial \rho v u}{\partial x} + \frac{\partial \rho v^2}{\partial y} + \frac{\partial \rho v w}{\partial z} + f\rho u + \frac{\rho(u^2 - v^2)\tan\phi}{a} + \frac{\partial p}{\partial y} = 0.$$

The atmosphere is assumed to be a perfect gas:

$$p = \mathcal{R}\rho T,$$

where $\mathcal{R}$ is the gas constant for dry air. Using this 'equation of state', the thermodynamic equation may be written

$$\frac{1}{\gamma p}\left(\frac{dp}{dt}\right) - \frac{1}{\rho}\left(\frac{d\rho}{dt}\right) = 0$$

where $\gamma = c_p/c_v$ is the ratio of specific heats.

3.2 "Finding the vertical velocity"

The vertical component of velocity in the atmosphere is typically two or three orders of magnitude smaller than the horizontal components. It is difficult to measure w and in general no observations of this variable are available. In particular, Richardson had no such observations for 0700 UTC on 20 May, 1910. Moreover, even if he had had such observations, he recognized the practical impossibility of computing the tendency $\partial w/\partial t$ which would have to be calculated as a tiny residual term in the vertical dynamical equation.

Richardson acknowledged the influence of Vilhelm Bjerknes' publications *Statics* and *Dynamics* (Bjerknes, 1910, 1911) on his work. In his Preface (LFR, p viii; Dover Edition, p xii) Richardson states that his choice of 'conventional strata', his use of specific momentum rather than velocity, his method of calculating vertical motion at ground level and his adoption of the hydrostatic approximation are all in accordance with Bjerknes' ideas.

The hydrostatic equation results from neglecting the vertical acceleration, and other small terms, in the vertical dynamical equation. But this precludes the possibility of calculating the acceleration $\partial w/\partial t$ directly. It was a stroke of genius for Richardson not only to realize the need to evaluate w diagnostically from the other fields but also to construct a magnificent mathematical equation to achieve this.

To construct Richardson's w-equation we eliminate the time dependency between the continuity equation and the thermodynamic equation using the hydrostatic equation.

Recall that the thermodynamic equation can be written in the form

$$\frac{1}{\gamma p}\left(\frac{\partial p}{\partial t} + \mathbf{V}\cdot\nabla p + w\frac{\partial p}{\partial z}\right) - \frac{1}{\rho}\frac{d\rho}{dt} = 0\,,$$

and that one of the various forms of the continuity equation is

$$\frac{1}{\rho}\frac{d\rho}{dt} + \left(\nabla\cdot\mathbf{V} + \frac{\partial w}{\partial z}\right) = 0.$$

We can eliminate the density between these and use the vertically integrated hydrostatic equation to get

$$\frac{1}{\gamma p}\left(-\int_z^\infty g\nabla\cdot\rho\mathbf{V}\,dz + \mathbf{V}\cdot\nabla p\right) + \left(\nabla\cdot\mathbf{V} + \frac{\partial w}{\partial z}\right) = 0.$$

Expanding the integrand and using the hydrostatic equation again we get

$$-\frac{1}{\gamma p}\int_z^\infty \left(g\rho\nabla\cdot\mathbf{V} + \frac{\partial\mathbf{V}}{\partial z}\cdot\nabla p\right)\,dz + \left(\nabla\cdot\mathbf{V} + \frac{\partial w}{\partial z}\right) = 0.$$

Since the upper limit of the integral is infinite, it is convenient to use pressure as the independent variable; this is done by using the hydrostatic equation once more, yielding the result:

$$\frac{\partial w}{\partial z} = -\nabla\cdot\mathbf{V} + \frac{1}{\gamma p}\int_0^p \left(\nabla\cdot\mathbf{V} - \frac{\partial\mathbf{V}}{\partial p}\cdot\nabla p\right)\,dp. \qquad (1)$$

This corresponds to (9) on page 124 of LFR, save that we have omitted the effects of moisture and diabatic forcing which were included by Richardson.

The solution of (1) for w is straightforward. The gradient $\partial w/\partial z$ is calculated for each layer, working downwards from the stratosphere since the integral vanishes at $p = 0$. Then w may be calculated at the interface of each layer, working upwards, once it is known at the Earth's surface. Richardson followed Bjerknes in taking the surface value

$$w_{\mathrm{S}} = (\mathbf{v}\cdot\nabla h)_{\mathrm{S}}\ .$$

This is equivalent to the kinematic condition that the ground is impervious to the wind. However, Richardson does not state how he evaluates $\mathbf{v}_{\mathrm{S}}$; in repeating his forecast we have assumed a simple relationship $\mathbf{v}_{\mathrm{S}} = k\mathbf{v}_5$ where $\mathbf{v}_5$ is the lowest layer velocity and $k = 0.2$.

The vertical velocity equation was a major contribution by Richardson to dynamic meteorology. In recognizing its essential rôle in his forecast scheme he observed (LFR, p178) that 'it might be called the keystone of the whole system, as so many other equations remain incomplete until the vertical velocity has been inserted'.

3.3 Temperature in the Stratosphere

Richardson devoted a full chapter of 24 pages to the stratosphere. We shall not discuss the bulk of this, but we must consider the means by which the temperature of the uppermost layer is forecast. For, in the scheme adopted by Richardson, the vertical integral of pressure through the stratospheric layer depends on the temperature so that prediction of the latter is essential to ensure a 'lattice-reproducing' scheme — that is, an algorithm which, starting with a set of variables at one instant, produces the corresponding set at a later instant.

Richardson calculated the change in stratospheric temperature using two different equations, his elaborate equation (8) on page 147 of LFR and a much simpler equation corresponding to (14) on page 143. The resulting temperature tendencies, given in his Computing Form P_{XIV} on page 201, were $9.1 \times 10^{-4}\,\mathrm{K\,s^{-1}}$ for the elaborate equation and $9.2 \times 10^{-4}\,\mathrm{K\,s^{-1}}$ for the simpler. In view of this close agreement, we shall confine attention to the simpler alternative

$$\frac{\partial T}{\partial t} = T\frac{\partial w}{\partial z}\,. \tag{2}$$

This equation is sufficient for predicting the stratospheric temperature as long as the assumptions of geostrophy and vertical isothermy are acceptable. We shall use this simple prognostic equation in the sequel.

4. "THE ARRANGEMENT OF POINTS AND INSTANTS"

Richardson chose to divide the atmosphere into five strata of approximately equal mass, separated by horizontal surfaces at 2.0, 4.2, 7.2 and 11.8 km, corresponding to the mean heights (over Europe) of the 800, 600, 400 and 200 hPa surfaces. He discusses this choice in LFR, §3/2. It is desirable to have a surface near the tropopause, one stratum for the planetary boundary layer and at least two more for the troposphere above the boundary layer. Taking layers of equal mass simplifies the treatment of processes like radiation, and the particular choice of surfaces at approximately 2, 4, 6 and 8 decibars greatly facilitates the extraction of initial data from the charts and tables of Bjerknes. The strata are depicted in Fig. 1. Each horizontal layer is divided up into rectangular boxes or grid-cells. Richardson selected boxes with sides of length $\Delta\lambda = 3^\circ$ in the East-West direction and 200 km (or $\Delta\phi = 1.8^\circ$) in the North-South direction.

The numerical integration of the equations is carried out by a step-by-step procedure — an algorithm — which produces later values from earlier ones. Richardson took pains to devise a numerical scheme such that *where a particular variable was given at an initial time, the corresponding value at a later time at the same point* could be computed. His scheme is best illustrated for the linear shallow water equations:

$$\frac{\partial U}{\partial t} - fV + \frac{\partial P}{\partial x} = 0$$

$$\frac{\partial V}{\partial t} + fU + \frac{\partial P}{\partial y} = 0$$

$$\frac{\partial P}{\partial t} + gH\nabla\cdot\mathbf{U} = 0\,.$$

The tendency of each component of momentum depends on the other momentum component and on the gradient of pressure. Thus, U and V should be specified at the same points, and these points should be inter-meshed with those where P is given:

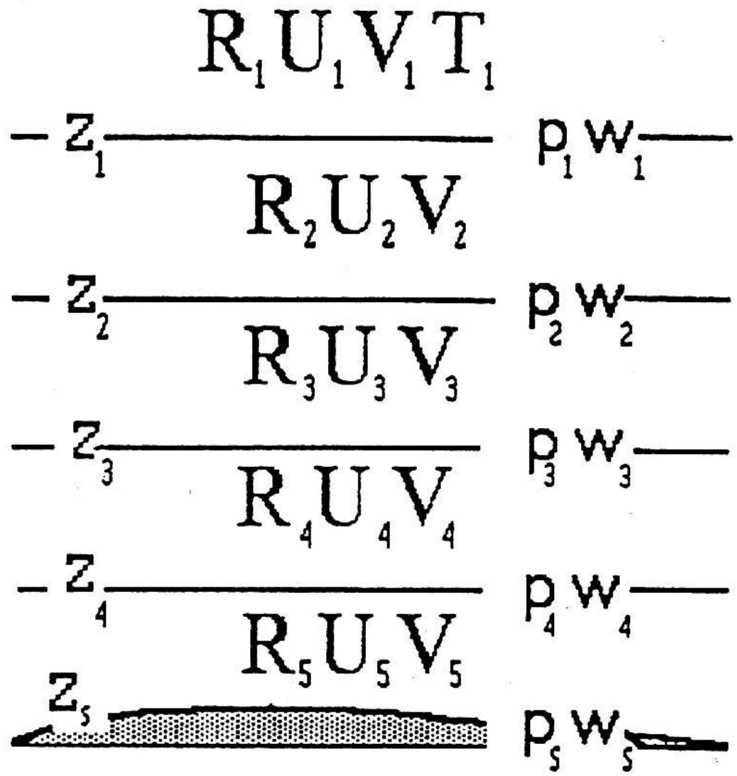

Fig. 1. Vertical stratification

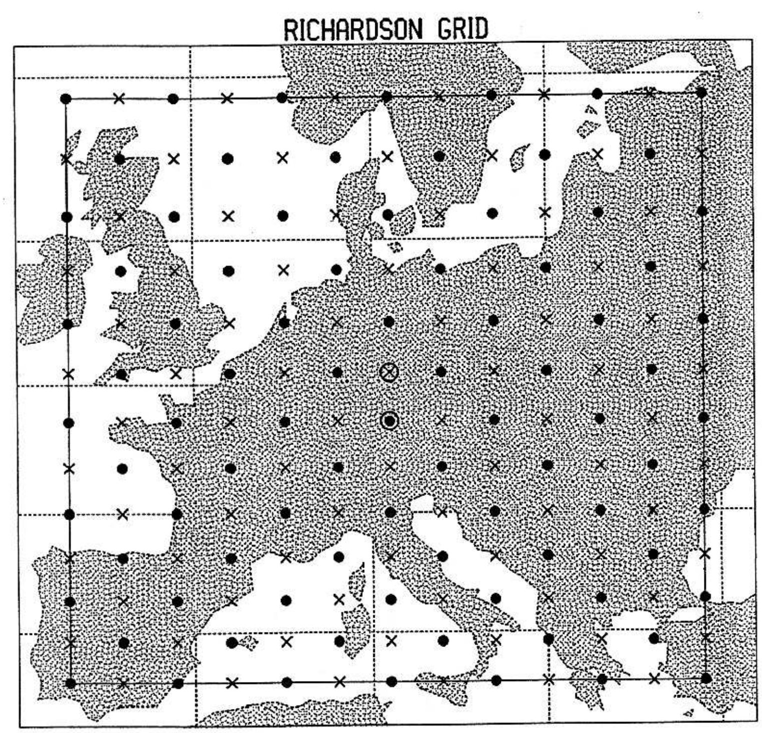

Fig. 2. Horizontal grid and geographical coverage.

U, V	P	U, V
P	U, V	P
U, V	P	U, V

This arrangement is also ideal for the continuity equation: for the tendency of pressure P depends on the divergence of momentum which is (primarily) comprised of horizontal derivatives of momentum, so P should be evaluated at points intermediate between those where momentum is given. This staggered arrangement of points is known today as an E-grid. Platzman's proposal to call it a Richardson grid has much to commend it (Platzman, 1967). When the values of a variable are specified on the discrete grid, spatial derivatives may be calculated approximately by means of finite differences. For example, the derivatives of P are, to second order accuracy,

$$\left(\frac{\partial P}{\partial x}\right)_{\mathrm{ij}} = \left(\frac{P_{\mathrm{i+1,j}} - P_{\mathrm{i-1,j}}}{2a\cos\phi_{\mathrm{j}}\,\Delta\lambda}\right), \qquad \left(\frac{\partial P}{\partial y}\right)_{\mathrm{ij}} = \left(\frac{P_{\mathrm{i,j+1}} - P_{\mathrm{i,j-1}}}{2a\Delta\phi}\right),$$

where $P_{\mathrm{i,j}}$ is the value of P at the point $(i\Delta\lambda, j\Delta\phi)$.

The geographical coverage used in repeating Richardson's forecast is shown in Fig. 2. P-points are indicated by solid circles and M-points by crosses. The region was chosen to best fulfil conflicting requirements: that it be as large as possible; that data coverage over the area be adequate; and that the points used by Richardson be located centrally in the region. The absence of observations precluded the extension of the region beyond that shown. The P-point and M-point for which Richardson calculated his tendencies are encircled.

The method of calculating the dependent quantities at each new time-level will now be described. It is performed by means of the familiar leapfrog scheme, called by Richardson the step-over method (LFR, p150). The prognostic variables are R, the mass per unit area and U and V, the components of momentum per unit area, for each stratum, and the temperature T_1 of the stratosphere. Let Q denote a typical dependent variable; it is governed by an equation of the form

$$\frac{\partial Q}{\partial t} = F$$

where F, the tendency of Q, is a function of Q and the other dependent variables. Let us assume that all the dependent variables are known at time $t = n\Delta t$ so that $F^n = F(n\Delta t)$ can be computed, and that the value of Q at the previous time level $t = (n-1)\Delta t$ has been retained. Then the forecast value Q^{n+1} may be computed from the old value Q^{n-1} and the tendency F^n:

$$Q^{n+1} = Q^{n-1} + 2\Delta t\, F^n .$$

The first step forward cannot be made with the leapfrog scheme, since the variables are known only at $t = 0$. A simple non-centered step

$$Q^1 = Q^0 + \Delta t \, F^0$$

provides values of the variables at $t = \Delta t$; from then on, the leapfrog scheme can be used.

The calculations of Richardson were confined to the evaluation of the initial tendencies (at 0700 on 20 May, 1910). He multiplied these by a time interval $2\Delta t = 6\,\mathrm{h}$ to represent the change over the six-hour period centered at 0700 UTC. In modern terminology, the time-step is specified as the interval between adjacent evaluations of the variables; thus, the time-step used by Richardson was three hours, not six hours as so often stated. A three-hour step was also chosen by him in describing his fantastic forecast factory (LFR, p219).

5. THE EQUATIONS FOR THE STRATA

As we have seen, Richardson divided the atmosphere into five 'conventional strata' separated by horizontal surfaces at fixed heights 2.0, 4.2, 7.2 and 11.8 km. These heights will be denoted respectively by z_1, z_2, z_3 and z_4, all constants. The variable height of the Earth's surface will be written $z_5 = h(\lambda, \phi)$. Variables at these five levels will be denoted by corresponding indices 1–5. Where convenient, values at the surface of the Earth may be indicated by subscript S.

The equations of motion will be integrated with respect to height across each stratum, to obtain expressions applying to the stratum as a whole. Quantities derived by integrating in this way will be denoted by capitals:

$$R = \int \rho \,\mathrm{d}z \quad P = \int p \,\mathrm{d}z \quad U = \int \rho u \,\mathrm{d}z \quad V = \int \rho v \,\mathrm{d}z \,.$$

The stratum is specified by the index corresponding to the *lower* level; thus, for example,

$$R_3 = \int_{z_3}^{z_2} \rho \,\mathrm{d}z \,.$$

In differentiating mean values for the lowest layer, allowance must be made for the variation of the height h of the Earth's surface. For the other layers, the limits are independent of x and y.

The continuity equation will now be integrated in the vertical. Taking, for example, the stratum between z_3 and z_2, and using the definitions of R, U and V, we get

$$\frac{\partial R_3}{\partial t} + \frac{\partial U_3}{\partial x} + \frac{\partial V_3}{\partial y} - \frac{V_3 \tan\phi}{a} + \big[\rho w\big]_2 - \big[\rho w\big]_3 = 0. \qquad (3)$$

The equations for the other upper layers are of similar form. For the lowest layer the slope of the bottom boundary must be considered, and the term $-\big[\rho w\big]_\mathrm{S}$ is cancelled by a term $[\rho \mathbf{v} \cdot \nabla h]_\mathrm{S}$.

The vertical integration of the horizontal equations of motion is performed in the same manner. To express the result in terms of the variables R, U and V it is necessary to make an approximation in the horizontal flux terms (see LFR, p 34). With this approximation the equations for the stratum (z_3, z_2) are

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}\left(\frac{U^2}{R}\right) + \frac{\partial}{\partial y}\left(\frac{UV}{R}\right) + [\rho uw]_2 - [\rho uw]_3$$

$$- \left(f + \frac{2U \tan\phi}{aR}\right) V + \frac{\partial P}{\partial x} = 0 \tag{4}$$

$$\frac{\partial V}{\partial t} + \frac{\partial}{\partial x}\left(\frac{UV}{R}\right) + \frac{\partial}{\partial y}\left(\frac{V^2}{R}\right) + [\rho vw]_2 - [\rho vw]_3$$

$$+fU + \frac{\left(U^2 - V^2\right)\tan\phi}{aR} + \frac{\partial P}{\partial y} = 0\ . \tag{5}$$

The equations for the other upper layers are of similar form. For the lowest layer the slope of the bottom boundary must be considered.

6. "REVIEW OF OPERATIONS IN SEQUENCE"

The title of this section is identical to that of LFR, Ch. VIII, in which Richardson traces, step by step, the sequence of calculations necessary to carry his forecast forward in time. Let us assume that all the dependent variables are known at time $t = n\Delta t$. The advancement to the next time level, $t = (n+1)\Delta t$, requires both prognostic and diagnostic components (these terms, borrowed from medicine, were introduced by Vilhelm Bjerknes). The prognostic variables are (at P-points) the mass per unit area R in each stratum and the stratospheric temperature T_1, and (at M-points) the components U and V of momentum in each stratum. Once these quantities are known for a particular moment, all the auxiliary fields (temperature, divergence, vertical velocity, etc.) for that moment can be calculated from diagnostic relationships.

The time-stepping calculations are done in a large loop which is repeated as often as required to reach the forecast span. The sequence of calculations will now be given. For each step, the number of the relevant Computing Form in LFR is indicated [in brackets]. First we consider the P-points.

1. [$\mathrm{P_I}$] The layer integral of pressure is calculated:

$$P = \Delta z \cdot \Delta p / \Delta \log p$$

Here Δ represents the difference in value across the layer. For the top layer $P = \mathcal{R}T_1/g$. The density integral is also calculated from

$$R = \frac{\Delta p}{g}.$$

2. [$\mathrm{P_I}$] Mean values for each stratum are calculated for various quantities, *e.g.*,

$$\bar{p} = \frac{P}{\Delta z} \qquad \bar{\rho} = \frac{R}{\Delta z} \qquad \bar{T} = \frac{\bar{p}}{\mathcal{R}\bar{\rho}}$$

3. [$\mathrm{P_{XIII}}$] The divergence of momentum $\nabla \cdot \mathbf{U}$ is computed for each level.

4. [$\mathrm{P_{XIII}}$] The values of $\nabla \cdot \mathbf{U}$ in the column above each P-point are summed up and the total multiplied by $-g$ to give the surface pressure tendency

$$\frac{\partial p_S}{\partial t} = -g \sum_{\text{all}}^{\text{strata}} \nabla \cdot \mathbf{U}$$

5. [$\mathrm{P_{XV}}$] The divergence of velocity $\delta = \nabla \cdot \mathbf{V}$ is calculated using the following approximations for mean velocity in each layer:

$$u = U/R \qquad v = V/R\,.$$

6. [$\mathrm{P_{XVI}}$] The vertical velocity gradient $\partial w/\partial z$ in each layer is now calculated using (1). The vertical velocity at the surface is determined from

$$w_S = \mathbf{v}_S \cdot \nabla h$$

where we assume $\mathbf{v}_S = k\mathbf{v}_5/R_5$ with $k = 0.2$. Then it is a straightforward matter to calculate w at each interface, working upward from the bottom.

7. [$\mathrm{P_{XIV}}$] The tendency of the stratospheric temperature T_1 is calculated next, using (2).

8. [$\mathrm{P_{XVI}}$] The temperature at each interface is calculated by linear interpolation. Then the density there is computed using the gas law, after which the momentum ρw at each interface can be obtained.

9. [$\mathrm{P_{XIII}}$] The tendency of the density integral R is now obtained using the continuity equation (3).

10. [$\mathrm{P_{XIII}}$] The final calculation at P-points is the tendency of pressure at each interface, obtained from

$$\left(\frac{\partial p}{\partial t}\right)_K = g \sum_{k=1}^{K} \left(\frac{\partial R_k}{\partial t}\right).$$

The surface pressure tendency, already computed in step 4, is confirmed here.

This completes the calculations required at the P-points. We now list the operations at the M-points [LFR, Computing Forms $\mathrm{M_{III}}$ and $\mathrm{M_{IV}}$]

11. The pressure gradient is evaluated by calculating the spatial derivatives of the integrated pressure P. For the lowest stratum there is an extra term due to orography. The x-component is given by

$$\frac{1}{2\Delta x}\left\{\delta P + \frac{\delta h \cdot \delta p_S}{\delta \log p_S}\right\}$$

where δ here represents the difference across a distance $2\Delta x$. The y-component is analogous.

12. The Coriolis terms and those involving $\tan\phi$ are evaluated. All the necessary quantities are available at the relevant points.

13. The horizontal flux terms are calculated. It is necessary to approximate the derivatives by differences over a distance $4\Delta x$ or $4\Delta y$.

14. The vertical flux terms are calculated. The momentum flux above the uppermost layer is assumed to vanish.

15. The tendencies of momenta, $\partial U/\partial t$ and $\partial V/\partial t$ may now be calculated, as all the other terms in (4) and (5) are known.

The tendencies of all prognostic variables are now known, and it is possible to update all the fields to the time $(n+1)\Delta t$. When this is done, the entire sequence of operations may be repeated in another time-step.

7. "AN EXAMPLE WORKED ON COMPUTING FORMS"

This section title refers to the elaborate set of 23 forms drawn up by Richardson for the arrangement of his calculations. They are included in LFR, pages 188–210, filled in with the values relevant to the two points to which his forecast applied. Richardson arranged, at his own expense, to have sets of blank forms printed so that they could be used by anyone wishing to carry out similar forecasts. I do not know if they were ever put to their intended use.

A computer program has been written to repeat and extend Richardson's forecast. The same initial values were used, so that the calculated initial changes could be compared directly with the values in LFR. It will be seen that the computer model produces results consistent with those obtained manually by Richardson. In particular, the "glaring error" in the surface pressure tendency is reproduced almost exactly by the model.

7.1 The Initial Tendencies

Richardson's computations were confined to the calculation of the initial tendencies at a single pair of points. These calculations amount to evaluating the right hand sides of equations of the form

$$\frac{\partial Q}{\partial t} = F.$$

The leapfrog method of integration in time amounts to approximating this equation by

$$\Delta Q = [Q(t+\Delta t) - Q(t-\Delta t)] = F(t) \times 2\Delta t\,.$$

It is important to note that the tendency F is independent of Δt, so that the change ΔQ is directly proportional to the time-step. The time-step between successive calculations is $\Delta t = 3\,\mathrm{h}$ and the changes given in LFR are over a six hour period centered at the initial time 0700 UTC 20 May 1910.

On p211 of LFR, Richardson presents his results for the changes in the prognostic variables at the two points. We first consider the changes of the pressures at each of the four interfaces and at the Earth's surface for the central P-point. The values obtained by Richardson are given in Table 3 (they are in the column marked LFR). The corresponding changes produced by the numerical model are also given, in the column marked MOD. The units of pressure change are hPa/6h. It is evident that the changes computed by the model are in close agreement with Richardson's calculations.

The changes of momentum calculated by Richardson and those computed with the model are given in Table 4. The agreement is not as close as for the pressure changes, but there is broad agreement between the two sets of forecast changes. The remaining prognostic variable is the temperature of the uppermost layer. The change tabulated by Richardson using (2) was $\Delta T_1 = 19.9°$. The computer model used the same equation and gave a forecast change of $\Delta T_1 = 19.6°$, in close agreement with Richardson. [2]

7.2 The source of the problem

Richardson ascribed the unrealistic value of pressure tendency to errors in the observed winds which resulted in spuriously large values of calculated divergence. This is true as far as it goes. However, the problem is deeper: even if the winds were modified to remove divergence completely at the initial time, large tendencies would soon be observed.

A subtle state of balance exists in the atmosphere between the pressure and wind fields, ensuring that the high frequency gravity waves have much smaller amplitude than the rotational part of the flow. Minor errors in observational data can result in a disruption of the balance, and cause large gravity wave oscillations in the model solution.

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[2] Was Richardsons $\Delta T_1 \approx 80°/\mathrm{day}$ the first-ever forecast of a stratospheric sudden warming?!!!

Table 3. Six-hour Changes in Pressure
Units: hPa/6h
LFR: Richardson; MOD: Model.

Level	LFR	MOD
1	48.3	48.5
2	77.0	76.7
3	103.2	102.1
4	126.5	124.5
S	145.1	145.4

Table 4. Six-hour Changes in Momentum Components
U: Eastward. V: Northward. Units: 10^3 kg/m/s/6h
LFR: Richardson; MOD: Model.

	U		V	
Layer	LFR	MOD	LFR	MOD
I	−73.0	−71.8	−33.7	−39.7
II	−19.6	−19.9	+23.8	+29.0
III	−8.9	−10.3	−13.8	−15.9
IV	−15.3	−13.7	−4.3	−4.1
V	−17.9	−22.5	+6.3	+7.1

Table 5: Six-hour Changes in Pressure Thickness
Richardson's Values: No Initialization

Layer	$(\partial\Delta p/\partial t)\Delta t$	Horizontal Convergence	Vertical Convergence
I	48.5	65.9	-17.4
II	28.4	-23.7	52.1
III	25.3	47.6	-22.3
IV	22.3	7.5	14.8
V	20.8	48.0	-27.2
Sum	145.4	145.4	0.0

Table 6: Six-hour Changes in Pressure Thickness
After Initialization by Digital Filter

Layer	$(\partial\Delta p/\partial t)\Delta t$	Horizontal Convergence	Vertical Convergence
I	-1.0	12.4	-13.4
II	-2.4	-33.3	30.8
III	-0.9	11.0	-11.9
IV	-0.5	-10.4	9.8
V	1.7	17.0	-15.4
Sum	-3.2	-3.2	0.0

They are avoided by modifying the data to restore harmony between the fields. We will describe a simple method of achieving balance, apply it to Richardson's data and show that it yields realistic results.

8. DIGITAL FILTER INITIALIZATION

To obtain reasonable values for the tendencies, we must reduce the high frequency components implicit in the initial data to realistic amplitudes. This process is called initialization. There are several ways to achieve it, one of the simplest being to use a digital filter. Such a filter was used by Lynch (1992) to initialize Richardson's barotropic data (see LFR, Ch. 2). We will apply the same technique below to the full baroclinic case.

Consider a function of time $f(t)$ with low and high frequency components. To filter out the high frequencies, we may proceed as follows:

(1) calculate the Fourier transform $F(\omega)$

(2) set coefficients of high frequencies to zero;

(3) calculate the inverse transform.

Step 2 may be performed by multiplying $F(\omega)$ by an appropriate weighting function $H(\omega)$. Typically, $H(\omega)$ is a step function, equal to one for $|\omega| \leq \omega_c$ and zero for $|\omega| > \omega_c$, with ω_c the cutoff frequency. The three steps are equivalent to a convolution:

$$f^*(t) = h * f(t) = \int_{-\infty}^{+\infty} h(t-\tau) f(\tau)\, d\tau,$$

where $h(t) = \sin(\omega_c t)/\pi t$ is the inverse Fourier transform of $H(\omega)$. To evaluate this integral approximately at $t = 0$, we calculate $f(t)$ at a finite set of times $\{-N\Delta t, \ldots, -\Delta t, 0, \Delta t, \ldots, N\Delta t\}$ and compute the sum

$$f^*(0) = \sum_{n=-N}^{N} f_n h_{-n}. \tag{6}$$

As is well known, truncation of a Fourier series may result in Gibbs oscillations. These may be greatly reduced by means of an appropriate window. The response is improved if h_n is modified by the Lanczos window $w_n = \sin[n\pi/N+1]/(n\pi/N+1)$.

The method outlined above was used to calculate filtered fields of height and wind at the initial time. The numerical model was integrated six hours forward and also six hours backward from the initial time, providing a sequence of values centered on $t = 0$ for each variable at each gridpoint. The cutoff was set at $\tau_c = 2\pi/\omega_c = 6\,\mathrm{h}$ and $\Delta t = 300\,\mathrm{s}$, so that $N = 72$. Filtered fields $f^*(t)$ could then be calculated using (6).

9. "SMOOTHING THE INITIAL DATA"

The idea of filtering in time goes right back to Richardson, who proposed several methods of smoothing the data, one way being to take the average value of observations made at successive times (LFR, Ch. X). The digital filtering method is similar, but the time series are generated by the model, and the filter is designed for optimal selectivity.

Fig. 3 shows the sea level pressure based on an extension of Richardson's values. The curious low near Strasbourg appears to be due to an error made by Richardson in converting sea level to surface pressure. This is confirmed by an examination of

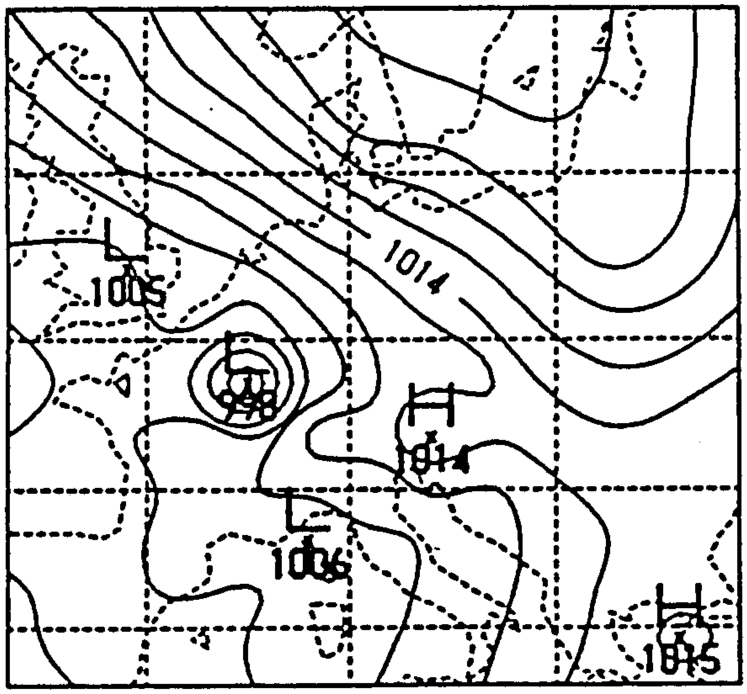

Fig. 3. Sea level Pressure. Original data.

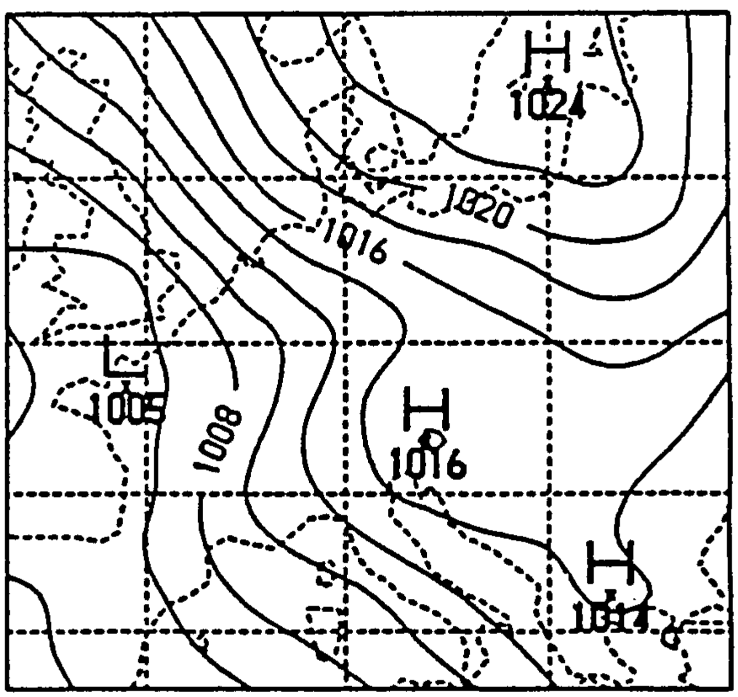

Fig. 4. Sea level Pressure. Filtered data.

the values in Table 2. The surface pressure at the point (48.6° N, 5.0° E) is seen to be suspiciously low. The pressure analysis after filtering is shown in Fig. 4. The changes induced by the initialization are seen to be small. However, notice the absence of the erroneous low near Strasbourg.

Platzman (1967) examined Richardson's results and discussed two problems contributing to the large pressure tendency: horizontal divergence values are too large, due to lack of cancellation between the terms; and there is a lack of compensation between convergence and divergence in the vertical. Table 5 shows the six-hour changes in pressure thickness for each level, and the contributions from the horizontal and vertical parts of divergence. The values are rather large, and there is little cancellation between them. Table 6 shows the corresponding figures for the filtered initial data. The values are all reduced, generally by about a factor of two. However, the changes are such that the compensation in the vertical between horizontal divergence at different levels is now much more complete. The result is that the surface pressure change is dramatically reduced in size, from 145 to 3 hPa — a realistic value. (The vertical convergence integrates to zero, making no contribution to surface pressure tendency). Clearly, the compensation in the vertical is vital in achieving balance.

10. CONCLUDING REMARKS

The numerical model was used to extend the forecast to 24 hours. It was found that spatial smoothing was required to maintain stability. Moreover, a timestep consistent with the CFL criterion was required. Thus, lack of initialization is not the only shortcoming of the method devised by Richardson. The results of the extended forecast will be reported elsewhere.

But let us suppose that Richardson had applied some filter, however crude, to his initial data. His results might well have been realistic, and his method would surely have been given the attention which it certainly deserved. For there can be little doubt that the failure of his trial forecast persuaded most meteorologists to ignore his work, so that his wonderful book gathered dust for many years. A more encouraging demonstration might have led his colleagues to consider his ideas more carefully and to investigate the potential usefulness of numerical forecasting in greater depth. [3] However, his fantastic forecast factory would hardly have come into being: even making no allowance for the short time step required for stability, his figure of 64,000 'computers' required to keep pace with the weather was a serious under-estimate (Lynch, 1993).

Richardson claimed that his prediction was 'a fairly correct deduction from a somewhat unnatural initial distribution'. The model results presented above confirm that he was fully justified in making this claim, and that what he presented in his book was indeed a marvellous forecast.

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[3] His venture in printing sets of blank forms might also have been more profitable.

References

Ashford, Oliver M., 1985: *Prophet—or Professor: The Life and Work of Lewis Fry Richardson.* Adam Hilger Ltd., Bristol. 303pp.

Bjerknes, V., 1904: Das Problem der Wettervorhersage, betrachtet vom Standpunkte der Mechanik und der Physik. *Meteor. Z.* **21**, 1–7.

Bjerknes, V. and J.W. Sandström, 1910: *Dynamic Meteorology and Hydrography.* Part I: Statics. Carnegie Institution, Washington, Publication 88 (Part I). 146pp and tables.

Bjerknes, V., Th. Hesselberg and O. Devik, 1911: *Dynamic Meteorology and Hydrography.* Part II: Kinematics. Carnegie Institution, Washington, Publication 88 (Part II). 175pp and Atlas of 60 Plates.

Bjerknes, V., 1914: *Veröffentlichungen des Geophysikalischen Instituts der Universität Leipzig.* Erste Serie: *Synoptische Darstellungen atmosphärischer Zustände.* Jahrgang 1910, Heft 3.

Lynch, Peter, 1992: Richardson's barotropic forecast: a reappraisal. *Bull. Amer. Met. Soc.*, **73**, 35–47.

Lynch, Peter, 1993: Richardson's forecast factory: the $64,000 question. *Met. Mag.*, **122**, 69–70.

Platzman, G.W., 1967: A retrospective view of Richardson's book on weather prediction. *Bull. Amer. Met. Soc.*, **48**, 514–550.

Richardson, L.F., 1922: *Weather Prediction by Numerical Process.* Cambridge Univ. Press, 236 pp. Reprinted by Dover Publications, New York, 1965.

TABLE CAPTIONS

FIGURE CAPTIONS