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Reference equations

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1 Introduction

In this manuscript we will try to derive conservative forms of the thermodynamic equation in the case of a complex micro-physical scheme including the ice-phase (i.e. we consider dry air (q_a), water vapour (q_v), liquid water (q_l), rain water (q_r), ice (q_i) and snow (q_s)). Liquid water and ice will remain in the particle and only rain water and snow will precipitate. This precipitation is represented by the partial mass flux $P_l + P_i = \rho_r w_r + \rho_s w_s$ with respect to the barycentre, where w_r and w_s are the partial vertical velocities of rain water and snow with respect to the barycentre. Next to this 'real' flux we also have the following pseudo fluxes: P_l' representing the integral of the transfer between vapour and liquid water due to condensation/evaporation; P_l'' representing the integral of the transfer between liquid and rainwater due to auto-conversion; P_l''' representing the integral of the transfer between rainwater and the water vapour due to evaporation of the falling liquid precipitation; P_i' representing the integral of the transfer between vapour and ice due to freezing/sublimation; P_i'' representing the integral of the transfer between ice and snow due to auto-conversion; P_i''' representing the integral of the transfer between snow and the water vapour due to sublimation of the falling solid precipitation. So we have (in a star shape):

$\boxed{q_i}$ $\quad P_i''$ (auto) $\quad \boxed{q_s}$

$\downarrow$

P_i' (freez) $\quad P_i'''$ (subli) $\quad P_i$ (snow)

$\boxed{q_v}$

P_l' (cond) $\quad P_l'''$ (evapor)

$\boxed{q_l}$ $\quad P_l''$ (auto) $\quad \boxed{q_r}$

$\downarrow$

P_l (rain)

Note that the process of melting/freezing between solid and liquid phases is considered such that the water goes through the vapour phase. Of course this is physically not the case but thermodynamically it is fully correct.

2 Case $\boldsymbol{\delta_m = 0}$

In this case any mass flux due to the motion of moisture is compensated by a flux of dry air. Following Martina's proposal for the barycentric case (i.e. that only dry air moves to compensate for the mass fall associated with precipitation), the conservation of the different mass species can be written as:

$$\frac{dq_v}{dt} = -g\frac{\partial}{\partial p}(P_l' + P_i') + g\frac{\partial}{\partial p}(P_l''' + P_i''') - g\frac{\partial J_{q_v}}{\partial p} \qquad (1)$$

$$\frac{dq_l}{dt} = +g\frac{\partial P_l'}{\partial p} - g\frac{\partial P_l''}{\partial p} - g\frac{\partial J_{q_l}}{\partial p} \tag{2}$$

$$\frac{dq_r}{dt} = +g\frac{\partial P_l''}{\partial p} - g\frac{\partial P_l'''}{\partial p} - g\frac{\partial P_l}{\partial p} \tag{3}$$

$$\frac{dq_i}{dt} = +g\frac{\partial P_i'}{\partial p} - g\frac{\partial P_i''}{\partial p} - g\frac{\partial J_{q_i}}{\partial p} \tag{4}$$

$$\frac{dq_s}{dt} = +g\frac{\partial P_i''}{\partial p} - g\frac{\partial P_i'''}{\partial p} - g\frac{\partial P_i}{\partial p} \tag{5}$$

$$\frac{dq_a}{dt} = +g\frac{\partial P_l}{\partial p} + g\frac{\partial P_i}{\partial p} - g\frac{\partial J_{q_a}}{\partial p} \tag{6}$$

with $J_{q_v}, J_{q_l}, J_{q_i}$ and J_{q_a} the respective turbulent fluxes such that $J_{q_v} + J_{q_l} + J_{q_i} + J_{q_a} = 0$. We can write the *local* thermodynamic equation (i.e. considering only the physical tendencies, hence the subscript Φ) as:

$$c_p\left(\frac{\partial T}{\partial t}\right)_\Phi = gL_l(T)\left(\frac{\partial P_l'}{\partial p} - \frac{\partial P_l'''}{\partial p}\right) + gL_i(T)\left(\frac{\partial P_i'}{\partial p} - \frac{\partial P_i'''}{\partial p}\right) - gP_l(c_l - c_{pd})\frac{\partial T}{\partial p} - gP_i(c_i - c_{pd})\frac{\partial T}{\partial p}$$
$$-g\frac{\partial J_s}{\partial p} + gT\left(c_{pd}\frac{\partial J_{q_a}}{\partial p} + c_{pv}\frac{\partial J_{q_v}}{\partial p} + c_l\frac{\partial J_{q_l}}{\partial p} + c_i\frac{\partial J_{q_i}}{\partial p}\right) - g\frac{\partial J_{rad}}{\partial p}$$

where, with respect to the current ARPEGE/ALADIN situation, we introduced in c_p the rainwater and snow fraction of the airmass and in the RHS the usage of latent heat by the corresponding new phase changes. J_s is the thermodynamic diffusive flux which doesn't contain any contribution of precipitation and J_{rad} is the radiative flux. Writing c_p as $c_p = c_{pd}q_a + c_{pv}q_v + c_l(q_l + q_r) + c_i(q_i + q_s)$ and using $L_{l|i}(T) = L_{l|i}(T=0) + (c_{pv} - c_{l|i})T$, we arrive at the following thermodynamic equation:

$$\left(\frac{\partial}{\partial t}(c_pT)\right)_\Phi = g\frac{\partial}{\partial p}\big[L_l(T)(P_l' - P_l''') - (c_{pv} - c_l)T(P_l' - P_l''') - (c_l - c_{pd})P_lT$$
$$+L_i(T)(P_i' - P_i''') - (c_{pv} - c_i)T(P_i' - P_i''') - (c_i - c_{pd})P_iT - J_s - J_{rad}\big], \tag{7}$$

which can be written alternatively

$$\left(\frac{\partial}{\partial t}(c_pT)\right)_\Phi = g\frac{\partial}{\partial p}\big[L_l(T=0)(P_l' - P_l''') + L_i(T=0)(P_i' - P_i''')$$
$$-(c_l - c_{pd})P_lT - (c_i - c_{pd})P_iT - J_s - J_{rad}\big].$$

This thermodynamic equation is a nice conservation law similar to the one without the explicit rain water and snow part (ARPEGE/ALADIN) but with the additional pseudo fluxes. The auto-conversion has of course no thermodynamic contribution.

3 Case $\delta_m = 1$

In this case the precipitation flux is not compensated by dry air any more. Nonetheless, new fluxes will appear due to the barycentric behaviour. As this case has also some dynamical consequences (surface pressure changes and so on) we should reverify them as well.

If we take a look at the surface and consider an evaporation flux E, a liquid precipitation flux R and a solid precipitation flux S (all positive downwards) we have the following advective and diffusive fluxes:

	$\delta_m = 0$ advective	diffusive	$\delta_m = 1$ advective	diffusive
q_a	0	$-g(E+R+S)$	$g(E+R+S)q_a$	$-g(E+R+S)q_a$
q_v	0	gE	$g(E+R+S)q_v$	$gE-g(E+R+S)q_v$
q_l	0	0	$g(E+R+S)q_l$	$-g(E+R+S)q_l$
q_r	0	gR	$g(E+R+S)q_r$	$gR-g(E+R+S)q_r$
q_i	0	0	$g(E+R+S)q_i$	$-g(E+R+S)q_i$
q_s	0	gS	$g(E+R+S)q_s$	$gS-g(E+R+S)q_s$

It is clear that the sum of the diffusive fluxes is in both cases zero and at the surface we have the advective flux $\delta_m g(E+R+S)$ and there is no total flux of q_a only when $\delta_m = 1$ and it is always the case for q_l and q_i. However we do have the problem that there is no continuation of the diffusive fluxes linked to precipitation in the atmosphere (in eqs. (1-6) and (10-15) there are no fluxes J_{q_r} and J_{q_s}), which gives us a boundary condition problem that may have to be treated separately. One could for instance have a constant flux throughout the vertical which has a zero divergence, but suggestions are welcome...

3.1 Continuity equation and consequences

The continuity equation becomes (there are no mass fluxes acting as source terms in the barycentric case):

$$\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial \eta}\right) = -\nabla.\left(\vec{v}\frac{\partial p}{\partial \eta}\right) - \frac{\partial}{\partial \eta}\left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)$$

The vertical velocities at the upper and lower boundaries are:

$$\eta = 0 \qquad \dot{\eta}\frac{\partial p}{\partial \eta} = 0$$

$$\eta = 1 \qquad \dot{\eta}\frac{\partial p}{\partial \eta} = \delta_m g(E+R+S)$$

Integrating the continuity equation above over $\eta = 0 \rightarrow 1$ and using the boundary conditions, the surface pressure tendency can be written as

$$\frac{\partial \pi_s}{\partial t} = -\int_0^1 \nabla.\left(\vec{v}\frac{\partial p}{\partial \eta}\right) d\eta - \delta_m g(E+R+S). \tag{8}$$

Using $B(\eta)$ as the proportionality factor of the surface pressure for the computation of the pressure along the vertical in the hybrid pressure-type coordinate system, $B'(\eta)$ as its derivative with respect to the generalised vertical coordinate η and substituting

$$\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial \eta}\right) = B'(\eta)\frac{\partial \pi_s}{\partial t},$$

in the continuity equation and integrating to level η we find

$$B(\eta)\frac{\partial \pi_s}{\partial t} = -\int_0^\eta \nabla.\left(\vec{v}\frac{\partial p}{\partial \eta}\right) d\eta - \dot{\eta}\frac{\partial p}{\partial \eta}.$$

The model coordinate-related vertical velocity can therefore be written as

$$\dot{\eta}\frac{\partial p}{\partial \eta} = B(\eta)\int_0^1 \nabla.\left(\vec{v}\frac{\partial p}{\partial \eta}\right) d\eta - \int_0^\eta \nabla.\left(\vec{v}\frac{\partial p}{\partial \eta}\right) d\eta + \delta_m g B(\eta)(E+R+S)$$

or

$$\left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)^{\delta_m=1} = \left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)^{\delta_m=0} + \delta_m g B(\eta)(E+R+S). \tag{9}$$

The pressure-related vertical velocity becomes

$$\begin{aligned}
\omega &= \frac{\partial p}{\partial t} + \vec{v}.\nabla p + \dot{\eta}\frac{\partial p}{\partial \eta} \\
&= \vec{v}.\nabla p + \int_0^\eta \frac{\partial}{\partial \eta}\left(\frac{\partial p}{\partial t} + \dot{\eta}\frac{\partial p}{\partial \eta}\right) d\eta \\
&= \vec{v}.\nabla p - \int_0^\eta \nabla.\left(\vec{v}\frac{\partial p}{\partial \eta}\right) d\eta,
\end{aligned}$$

which doesn't depend on the precipitation flux any more, thanks to the 'barycentric choice'.

3.2 Conservation of the different species

In the barycentric environment the precipitation flux will cause a compensating lift (referred to the barycentre) of the non-precipitating part (dry air, water vapour, liquid and solid water). The corresponding fluxes of the non-precipitating species should cancel out the total precipitating flux. Hence we can write for the water vapour part (and similar for the other non-precipitating species) the compensating flux as

$$\frac{q_v(P_l+P_i)}{q_a+q_v+q_l+q_i} = \frac{q_v(P_l+P_i)}{1-q_r-q_s}$$

Introducing directly δ_m with respect to eq. (1), the conservation of water vapour can be written as

$$\frac{\partial}{\partial t}\left(q_v\frac{\partial p}{\partial \eta}\right) = -\nabla.\left(q_v\vec{v}\frac{\partial p}{\partial \eta}\right) - \frac{\partial}{\partial \eta}\left(q_v\dot{\eta}\frac{\partial p}{\partial \eta}\right)$$

$$-g\frac{\partial}{\partial \eta}\left(P_l' - P_l''' + P_i' - P_i''' - \delta_m\frac{q_v(P_l+P_i)}{1-q_r-q_s}\right) - g\frac{\partial J_{q_v}}{\partial \eta}.$$

Subtracting $q_v\times$ the continuity equation and finally multiplying by $\partial\eta/\partial p$ gives us

$$\frac{dq_v}{dt} = -g\frac{\partial}{\partial p}(P_l'+P_i') + g\frac{\partial}{\partial p}(P_l'''+P_i''') + \delta_m g\frac{\partial}{\partial p}\left(q_v\frac{P_l+P_i}{1-q_r-q_s}\right) - g\frac{\partial J_{q_v}}{\partial p} \tag{10}$$

Similarly we have

$$\frac{dq_l}{dt} = +g\frac{\partial P_l'}{\partial p} - g\frac{\partial P_l''}{\partial p} + \delta_m g\frac{\partial}{\partial p}\left(q_l\frac{P_l+P_i}{1-q_r-q_s}\right) - g\frac{\partial J_{q_l}}{\partial p} \tag{11}$$

$$\frac{dq_r}{dt} = +g\frac{\partial P_l''}{\partial p} - g\frac{\partial P_l'''}{\partial p} - g\frac{\partial P_l}{\partial p} \tag{12}$$

$$\frac{dq_i}{dt} = +g\frac{\partial P_i'}{\partial p} - g\frac{\partial P_i''}{\partial p} + \delta_m g\frac{\partial}{\partial p}\left(q_i\frac{P_l+P_i}{1-q_r-q_s}\right) - g\frac{\partial J_{q_i}}{\partial p} \tag{13}$$

$$\frac{dq_s}{dt} = +g\frac{\partial P_i''}{\partial p} - g\frac{\partial P_i'''}{\partial p} - g\frac{\partial P_i}{\partial p} \tag{14}$$

$$\frac{dq_a}{dt} = (1-\delta_m)g\left(\frac{\partial P_l}{\partial p} + \frac{\partial P_i}{\partial p}\right) + \delta_m g\frac{\partial}{\partial p}\left(q_a\frac{P_l+P_i}{1-q_r-q_s}\right) - g\frac{\partial J_{q_a}}{\partial p} \tag{15}$$

It is rapidly verified that all terms cancel out, so $q_a+q_v+q_l+q_r+q_i+q_s$ remains 1. In case $\delta_m=0$, this set transforms to eqs. (1-6). Moreover, these expressions are fully consistent with the AROME equations, given by

$$\frac{dq_k}{dt} = -\frac{1}{\rho}\partial_\alpha(\rho q_k v_k^\alpha) + \frac{\dot{\rho}_k}{\rho} - \frac{1}{\rho}\partial_\alpha(\overline{\rho' q_k' v_k'^\alpha}), \qquad k = a, v, l, r, i, s \quad (\alpha = 3)$$

where the vertical velocity of the non-precipitating species with respect to the barycentre is

$$w_k = -\frac{P_l + P_i}{\rho(1 - q_r - q_s)} = -\frac{q_r w_r + q_s w_s}{1 - q_r - q_s}$$

and with $\overline{\rho' q_k' v_k'}$ the turbulent flux of the considered species, which is zero for precipitating species.

3.3 The Thermodynamic equation

For simplicity we will put back $\delta_m = 1$ in this paragraph.

As there are compensating enthalpy fluxes due to the barycentric regime, these fluxes will also have to be included in the thermodynamic equation:

$$\begin{aligned}\frac{\partial}{\partial t}\left(T\frac{\partial p}{\partial \eta}\right) &= -\nabla.\left(T\vec{v}\frac{\partial p}{\partial \eta}\right) - \frac{\partial}{\partial \eta}\left(T\dot{\eta}\frac{\partial p}{\partial \eta}\right) + \frac{g}{c_p}L_l(T)\left(\frac{\partial P_l'}{\partial \eta} - \frac{\partial P_l'''}{\partial \eta}\right) \\ &+\frac{g}{c_p}L_i(T)\left(\frac{\partial P_i'}{\partial \eta} - \frac{\partial P_i'''}{\partial \eta}\right) - \frac{g}{c_p}\left[c_l P_l + c_i P_i - \frac{c_p - c_l q_r - c_i q_s}{1 - q_r - q_s}(P_l + P_i)\right]\frac{\partial T}{\partial \eta} \\ &-\frac{g}{c_p}\frac{\partial J_s}{\partial \eta} + \frac{gT}{c_p}\left(c_{pd}\frac{\partial J_{q_a}}{\partial \eta} + c_{pv}\frac{\partial J_{q_v}}{\partial \eta} + c_l\frac{\partial J_{q_l}}{\partial \eta} + c_i\frac{\partial J_{q_i}}{\partial \eta}\right) - \frac{g}{c_p}\frac{\partial J_{rad}}{\partial \eta}\end{aligned} \tag{16}$$

Once again the physical part of this equation is fully consistent with the physical part of the AROME thermodynamic equation (for $\alpha = 3$):

$$c_p\left(\frac{\partial T}{\partial t}\right)_\Phi = \dot{Q}_i + \frac{1}{\rho}(L_{21}(\dot{\rho}_2 + \dot{\rho}_3) + L_{41}(\dot{\rho}_4 + \dot{\rho}_5)) - \sum_k c_{pk} q_k v_k^\alpha \partial_\alpha T, \qquad k = a, v, l, r, i, s$$

where $\dot{\rho}_2$ corresponds to P_l', $\dot{\rho}_3$ to $-P_l'''$, $\dot{\rho}_4$ corresponds to P_i', $\dot{\rho}_5$ to $-P_i'''$ and $\dot{Q}_i$ is the divergence of the diffusive heat flux. The vertical velocities are given in the previous subsection.

Rewriting the physical part of equation (16) gives us

$$\begin{aligned}c_p\left(\frac{\partial T}{\partial t}\right)_\Phi &= gL_l(T)\left(\frac{\partial P_l'}{\partial p} - \frac{\partial P_l'''}{\partial p}\right) + gL_i(T)\left(\frac{\partial P_i'}{\partial p} - \frac{\partial P_i'''}{\partial p}\right) - g\frac{\partial J_s}{\partial p} - g\frac{\partial J_{rad}}{\partial p} \\ -g\left[c_l P_l + c_i P_i - \frac{c_p - c_l q_r - c_i q_s}{1 - q_r - q_s}(P_l + P_i)\right]\frac{\partial T}{\partial p} &+ gT\left(c_{pd}\frac{\partial J_{q_a}}{\partial p} + c_{pv}\frac{\partial J_{q_v}}{\partial p} + c_l\frac{\partial J_{q_l}}{\partial p} + c_i\frac{\partial J_{q_i}}{\partial p}\right).\end{aligned} \tag{17}$$

Writing c_p as $c_p = c_{pd}q_a + c_{pv}q_v + c_l(q_l + q_r) + c_i(q_i + q_s)$ and using $L_{l|i}(T) = L_{l|i}(T = 0) + (c_{pv} - c_{l|i})T$, we can rewrite equation (17) as

$$\begin{aligned}\left(\frac{\partial}{\partial t}(c_p T)\right)_\Phi &= g\frac{\partial}{\partial p}\left[L_l(T)(P_l' - P_l''')\right] - (c_{pv} - c_l)\frac{\partial T}{\partial p}(P_l' - P_l''') \\ &+g\frac{\partial}{\partial p}\left[L_i(T)(P_i' - P_i''')\right] - (c_{pv} - c_i)\frac{\partial T}{\partial p}(P_i' - P_i''')\end{aligned}$$

$$-g\left[\,c_lP_l + c_iP_i - \frac{c_p - c_lq_r - c_iq_s}{1-q_r-q_s}(P_l+P_i)\right]\frac{\partial T}{\partial p} - g\frac{\partial J_{rad}}{\partial p}$$
$$-gT\frac{\partial}{\partial p}\left[\,c_lP_l + c_iP_i - \frac{c_p - c_lq_r - c_iq_s}{1-q_r-q_s}(P_l+P_i)\right] - g\frac{\partial J_s}{\partial p}$$
$$-gc_{pv}T\frac{\partial}{\partial p}(P_l' - P_l''' + P_i' - P_i''') + gc_lT\frac{\partial}{\partial p}(P_l' - P_l''') + gc_iT\frac{\partial}{\partial p}(P_i' - P_i''')$$

Finally we arrive at,

$$\left(\frac{\partial}{\partial t}(c_pT)\right)_\Phi = g\frac{\partial}{\partial p}\left[L_l(T)(P_l' - P_l''') - (c_{pv}-c_l)T(P_l' - P_l''') + L_i(T)(P_i' - P_i''') \right.$$
$$\left. -(c_{pv}-c_i)T(P_i' - P_i''') + (\hat{c}(P_l+P_i) - c_lP_l - c_iP_i)T - J_s - J_{rad}\right] \tag{18}$$

with

$$\hat{c} = \frac{c_{pd}q_a + c_{pv}q_v + c_lq_l + c_iq_i}{1-q_r-q_s}$$

3.4 $\delta_m = 0$ vs $\delta_m = 1$

The combination can be written as

$$\left(\frac{\partial}{\partial t}(c_pT)\right)_\Phi = g\frac{\partial}{\partial p}\left[L_l(T)(P_l' - P_l''') - (c_{pv}-c_l)T(P_l' - P_l''') - (c_l - c_{pd})P_lT \right.$$
$$+L_i(T)(P_i' - P_i''') - (c_{pv}-c_i)T(P_i' - P_i''') - (c_i - c_{pd})P_iT$$
$$\left. +\delta_m(\hat{c} - c_{pd})(P_l+P_i)T - J_s - J_{rad}\right] \tag{19}$$

Equation (19) together with eqs. (10-15) show the impact of the δ_m option in all closed budget equations of our set.

3.5 Simplification

Similarly as we rewrote eq. (7) we can simplify equation (19) to

$$\left(\frac{\partial}{\partial t}(c_pT)\right)_\Phi = -g\frac{\partial}{\partial p}[(c_l - c_{pd})P_lT + (c_i - c_{pd})P_iT - \delta_m(\hat{c} - c_{pd})(P_l+P_i)T]$$
$$+L_l(T=0)\left(\frac{\partial P_l'}{\partial p} - \frac{\partial P_l'''}{\partial p}\right) + L_i(T=0)\left(\frac{\partial P_i'}{\partial p} - \frac{\partial P_i'''}{\partial p}\right) - g\frac{\partial J_s}{\partial p} - g\frac{\partial J_{rad}}{\partial p} = -g\frac{\partial J_{total}}{\partial p} \tag{20}$$

with J_{total} a short-hand notation for showing that the whole right hand side of equation (20) is only a flux divergence and which we will frequently use in the next section. In this form of the equation we better see how the arbitrariness of $P_{l|i}'$ and $P_{l|i}'''$ (up to a constant) does not matter, since only the divergence of these quantities are used and since those divergences are only multiplied by constant values for the latent heat ($L_{l|i}(T=0)$).

4 Non-Hydrostatic Compressible Projection

All the above is only true when the addition/removal of heat is projected only on a temperature change and has no pressure change equivalent. This is of course no problem in the hydrostatic case but when going to the compressible case (where no anelastic approximation is made), we would like to include also the more physical option where any heat source is projected on both temperature and pressure changes.

For simplicity, the demonstration below will not include turbulent fluxes. It will implicitely be assumed that $\delta_m = 1$.

4.1 The Thermodynamic equation

We start with the general entropy expression given by Marquet which we multiply by q_a in order to have an expression per unit of mass:

$$S' = q_a S = (q_a c_{pd} + q_t c_{pv})\ln(T) - q_a R_d \ln(p-e) - q_t R_v \ln(e) - \frac{L_l(T)}{T}(q_l + q_r) - \frac{L_i(T)}{T}(q_i + q_s) \quad (21)$$

with $q_t = q_v + q_l + q_r + q_i + q_s$ and S' the total entropy per unit of mass. Note that this expression does not include any precipitation processes but those will be added later.

Time-derivation gives us:

$$\frac{dS'}{dt} = \frac{\widetilde{Q}}{T} = (q_a c_{pd} + q_t c_{pv})\frac{1}{T}\frac{dT}{dt} + c_{pd}\ln(T)\frac{dq_a}{dt} + c_{pv}\ln(T)\frac{dq_t}{dt} - R_d\ln(p-e)\frac{dq_a}{dt}$$
$$-\frac{q_a R_d}{p-e}\frac{d(p-e)}{dt} - R_v\ln(e)\frac{dq_t}{dt} - \frac{q_t R_v}{e}\frac{de}{dt} - \frac{L_l(T)}{T}\left(\frac{dq_l}{dt} + \frac{dq_r}{dt}\right) - \frac{L_i(T)}{T}\left(\frac{dq_i}{dt} + \frac{dq_s}{dt}\right)$$
$$-\frac{(c_{pv} - c_l)}{T}(q_l + q_r)\frac{dT}{dt} - \frac{(c_{pv} - c_i)}{T}(q_i + q_s)\frac{dT}{dt} + \frac{L_l(T)}{T^2}(q_l + q_r)\frac{dT}{dt} + \frac{L_i(T)}{T^2}(q_i + q_s)\frac{dT}{dt} \quad (22)$$

with $\widetilde{Q}$ the diabatic heat source out of which the precipitation effects are filtered. Following Bannon (2002), we subtract from the expression above the change in entropy due to precipitation. This change can be written as a sum of entropy-fluxes associated with the different mass-fluxes with respect to the barycentre. We have the following mass-fluxes:

$$\begin{aligned}
q_a &\rightarrow & F_a &= -q_a\frac{P_l + P_i}{1 - q_r - q_s}\\
q_v &\rightarrow & F_v &= -q_v\frac{P_l + P_i}{1 - q_r - q_s}\\
q_l &\rightarrow & F_l &= -q_l\frac{P_l + P_i}{1 - q_r - q_s}\\
q_r &\rightarrow & F_r &= P_l\\
q_i &\rightarrow & F_i &= -q_i\frac{P_l + P_i}{1 - q_r - q_s}\\
q_s &\rightarrow & F_s &= P_i
\end{aligned}$$

such that the associated entropy change can be written as

$$dS_{precip} = -g\frac{\partial}{\partial p}(s_a F_a) - g\frac{\partial}{\partial p}(s_v F_v) - g\frac{\partial}{\partial p}(s_l F_l) - g\frac{\partial}{\partial p}(s_r F_r) - g\frac{\partial}{\partial p}(s_i F_i) - g\frac{\partial}{\partial p}(s_s F_s) \quad (23)$$

with s_x the specific entropy of the different mass species which we can write as (up to a constant):

$$\begin{aligned}
s_a &= c_{pd}\ln(T) - R_d\ln(p-e)\\
s_v &= c_{pv}\ln(T) - R_v\ln(e)\\
s_l &= c_l\ln(T)\\
s_r &= c_l\ln(T)\\
s_i &= c_i\ln(T)\\
s_s &= c_i\ln(T)
\end{aligned}$$

Rewriting (23), we must subtract

$$+g c_{pd} q_a\frac{P_l + P_i}{1 - q_r - q_s}\frac{1}{T}\frac{\partial T}{\partial p} + g c_{pd}\ln(T)\frac{\partial}{\partial p}\left(q_a\frac{P_l + P_i}{1 - q_r - q_s}\right) - g R_d\ln(p-e)\frac{\partial}{\partial p}\left(q_a\frac{P_l + P_i}{1 - q_r - q_s}\right)$$

$$+gc_{pv}q_v\frac{P_l+P_i}{1-q_r-q_s}\frac{1}{T}\frac{\partial T}{\partial p}+gc_{pv}\ln(T)\frac{\partial}{\partial p}\left(q_v\frac{P_l+P_i}{1-q_r-q_s}\right)-gR_v\ln(e)\frac{\partial}{\partial p}\left(q_v\frac{P_l+P_i}{1-q_r-q_s}\right)$$
$$+gc_lq_l\frac{P_l+P_i}{1-q_r-q_s}\frac{1}{T}\frac{\partial T}{\partial p}+gc_l\ln(T)\frac{\partial}{\partial p}\left(q_l\frac{P_l+P_i}{1-q_r-q_s}\right)-gc_l\frac{P_l}{T}\frac{\partial T}{\partial p}-gc_l\ln(T)\frac{\partial P_l}{\partial p}$$
$$+gc_iq_i\frac{P_l+P_i}{1-q_r-q_s}\frac{1}{T}\frac{\partial T}{\partial p}+gc_i\ln(T)\frac{\partial}{\partial p}\left(q_i\frac{P_l+P_i}{1-q_r-q_s}\right)-gc_i\frac{P_i}{T}\frac{\partial T}{\partial p}-gc_i\ln(T)\frac{\partial P_i}{\partial p} \quad (24)$$

from (22). Doing so and using the following relations

$$q_v = q_a\frac{R_d}{R_v}\frac{e}{p-e} \quad (25)$$
$$\frac{de}{dT} = \frac{L_{l|i}(T)\rho_v}{T} \quad (26)$$
$$\frac{dq_a}{dt} = -\frac{dq_t}{dt} \quad (27)$$
$$L_l(T) = T(s_v-s_l) \quad \text{and} \quad L_i(T)=T(s_v-s_i) \quad (28)$$

we finally get

$$\widetilde{Q}=c_p\frac{dT}{dt}-\frac{1}{\rho}\frac{dp}{dt}-gL_l(T)\left(\frac{\partial P_l'}{\partial p}-\frac{\partial P_l'''}{\partial p}\right)-gL_i(T)\left(\frac{\partial P_i'}{\partial p}-\frac{\partial P_i'''}{\partial p}\right)$$
$$+\left[gc_lP_l+gc_iP_i-g\frac{c_p-c_lq_r-c_iq_s}{1-q_r-q_s}(P_l+P_i)\right]\frac{\partial T}{\partial p} \quad (29)$$

which we can rewrite as

$$c_p\frac{dT}{dt}-RT\frac{d\ln(p)}{dt}=Q, \quad (30)$$

with Q the full diabatic heat source. If we define this heat source as

$$Q=-g\frac{\partial J_{total}}{\partial p}-T\frac{dc_p}{dt}, \quad (31)$$

we can rewrite (29) as

$$\frac{d(c_pT)}{dt}-RT\frac{d\ln(p)}{dt}=-g\frac{\partial J_{total}}{\partial p}. \quad (32)$$

In case of no precipitation and phase-changes (the linear case, i.e. constant c_p and R in the lagrangian sense) (30) becomes

$$c_p\frac{dT}{dt}-RT\frac{d\ln(p)}{dt}=\overline{Q}, \quad (33)$$

with $\overline{Q}$ the 'linear' diabatic heat source which is the heat source out of which the effects of both phase-changes and precipitation are filtered out.

4.2 The Compressible case

In the compressible case we want to have a choice between the equivalent of the above (so-called quasi-anelastic approximation for the physical forcing, i.e. an unchanged thermodynamic equation with respect to the hydrostatic case as well as no pressure effect) and the more physical one where any heat source is projected on both temperature and pressure changes.

In the linear case (constant c_p, c_v and R) this means replacing

$$c_p \frac{dT}{dt} - RT \frac{d\ln(p)}{dt} = \overline{Q} \tag{34}$$
$$c_v \frac{d\ln(p)}{dt} + c_p D_3 = 0 \tag{35}$$

by

$$c_p \frac{dT}{dt} - RT \frac{d\ln(p)}{dt} = \overline{Q} \tag{36}$$
$$c_v \frac{d\ln(p)}{dt} + c_p D_3 = \frac{\overline{Q}}{T} \tag{37}$$

with $D_3 = -d\ln(\rho)/dt$ the three-dimensional divergence. Note that the first equation does not change in this way of writing. Eliminating the cross-term $d\ln(p)/dt$, the first lines of both sets of equations also write:

$$\frac{dT}{dt} + \frac{RT}{c_v} D_3 = \frac{\overline{Q}}{c_p} \tag{38}$$
$$\frac{dT}{dt} + \frac{RT}{c_v} D_3 = \frac{\overline{Q}}{c_v}, \tag{39}$$

which shows the change of emphasis from c_p to c_v if one has the same dynamical terms.

Leaving the linear case, we assume varying c_p, c_v and R and start with the accepted equation (30)

$$c_p \frac{dT}{dt} - RT \frac{d\ln(p)}{dt} = Q.$$

Using the definition of Q (eq. (31)), the state law $p = \rho RT$, the relation $c_p = c_v + R$ and the continuity equation in the following form

$$\frac{d\ln(\rho)}{dt} + D_3 = 0$$

we find

$$\frac{d(c_v T)}{dt} + RT\ D_3 = -g \frac{\partial J_{total}}{\partial p} \tag{40}$$

Using the time derivative of the logarithmic form of the state law, we also obtain

$$c_v \frac{d\ln(p)}{dt} + c_p D_3 = -\frac{g}{T} \frac{\partial J_{total}}{\partial p} - \frac{dc_v}{dt} + \frac{c_v}{R} \frac{dR}{dt} \tag{41}$$

So we finally have the following set of equations:

$$c_v \frac{dT}{dt} + RT\ D_3 = Q + T \frac{dR}{dt} \tag{42}$$
$$c_v \frac{d\ln(p)}{dt} + c_p D_3 = \frac{Q}{T} + \frac{c_p}{R} \frac{dR}{dt} \tag{43}$$

Equation (41) can also be written as

$$\frac{d\ln(c_v p/R)}{dt} + \frac{c_p}{c_v} D_3 = -\frac{g}{c_v T} \frac{\partial J_{total}}{\partial p} \tag{44}$$

4.3 Summary

In the general case (phase-changes and precipitation) we have in the hydrostatic and anelastic cases the following set of equations:

$$\begin{aligned} c_p \frac{dT}{dt} - RT \frac{d\ln(p)}{dt} &= Q \qquad (45) \\ c_v \frac{d\ln(p)}{dt} + c_p D_3 &= 0, \qquad (46) \end{aligned}$$

whereas in the compressible case we have the following option

$$\begin{aligned} c_v \frac{dT}{dt} + RT\ D_3 &= Q + T\frac{dR}{dt} \qquad (47) \\ c_v \frac{d\ln(p)}{dt} + c_p D_3 &= \frac{Q}{T} + \frac{c_p}{R}\frac{dR}{dt}. \qquad (48) \end{aligned}$$

Note that the first equation of both sets are equivalent. From the practical point of view this means replacing

$$\begin{aligned} \frac{d(c_p T)}{dt} - RT\frac{d\ln(p)}{dt} &= -g\frac{\partial J_{total}}{\partial p} \qquad (49) \\ c_v \frac{d\ln(p)}{dt} + c_p D_3 &= 0 \qquad (50) \end{aligned}$$

by

$$\begin{aligned} \frac{d(c_v T)}{dt} + RT\ D_3 &= -g\frac{\partial J_{total}}{\partial p} \qquad (51) \\ \frac{d\ln(c_v p/R)}{dt} + \frac{c_p}{c_v} D_3 &= -\frac{g}{c_v T}\frac{\partial J_{total}}{\partial p} \qquad (52) \end{aligned}$$

5 Remarks

5.1 Absolute vs relative motions

If we put $P_l^* = w_r^* \rho_r$ and $P_i^* = w_s^* \rho_s$ respectively the rain and snow flux as seen by the micro-physical scheme (the 'absolute' fluxes, with w_r^* and w_s^* the 'absolute' falling velocities), then we can write the corresponding fluxes with respect to the barycentre as

$$\begin{aligned} P_l &= w_r \rho_r = (1-q_r)P_l^* - q_r P_i^* \qquad (53) \\ P_i &= w_s \rho_s = (1-q_s)P_i^* - q_s P_l^*, \qquad (54) \end{aligned}$$

so that

$$P_l + P_i = (1 - q_r - q_s)(P_l + P_i)^*. \qquad (55)$$

Now one can rewrite the conservative form of the thermodynamic equation in function of the absolute fluxes or velocities.

5.2 Number of precipitation processes

It is easily verified that in case of only one precipitation process all the equations stated in this manuscript will transform to the equations of a single precipitation process. In case there are several precipitation processes associated with one of the water phases (like snow, graupel and hail, for example) the system can easily be expanded to cover distinctions in the fall velocity (the only parameter that matters for the type of equations written here).

5.3 Present ARPEGE/ALADIN case (conceptualised as: $P_{prec} = \rho_{prec} w_{prec}$ with $\rho_{prec} = 0$ and $w_{prec} = \infty$)

In this case the mass of precipitating species is neglected ($\rho_{prec} = q_r = q_s = 0$) but their vertical velocity of precipitation is infinite ($w_{prec} = w_r = w_s = \infty$) to allow the removal of the precipitation in one time-step.
The product of both quantities gives back the precipitation flux and once again it is possible to verify that the ARPEGE/ALADIN equations are a special case of equations (10-15) and (17), at least for the case $delta_m = 0$.